# Alternative technique for complex spectra analysis

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The choice of a suitable random matrix model of a complex system is very sensitive to the nature of its complexity. The statistical spectral analysis of various complex systems requires, therefore, a thorough probing of a wide range of random matrix ensembles which is not an easy task. It is highly desirable, if possible, to identify a common mathematical structure among all the ensembles and analyze it to gain information about the ensemble properties. Our successful search in this direction leads to the Calogero Hamiltonian, a one-dimensional quantum Hamiltonian with inverse-square interaction, as the common base. This is because both the eigenvalues of the ensembles and a general state of the Calogero Hamiltonian evolve in an analogous way for arbitrary initial conditions. The varying nature of the complexity is reflected in different forms of the evolution parameter in each case. A complete investigation of the Calogero Hamiltonian can then help us in the spectral analysis of complex systems.

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Recent statistical studies in various branches of theoretical physics, ranging from the Calogero model of a one-dimensional (1D) fermionic system [1], a random matrix (RM) model of disordered systems, and matrix models of random surfaces to a nonlinear $\sigma$ model (NLSM) of quantum chaotic systems, have revealed the presence of a common mathematical structure [2–4]. The connecting web of these various models with each other is well described in [3]. However, so far, the connection of a RM model with other models was established only for standard Gaussian ensembles (SGE), that is, Gaussian ensembles invariant under unitary transformation. This was achieved by showing that the distribution of the eigenvalues of the ensemble is governed by a Fokker-Planck (FP) equation [6,7] similar to that of Dyson's ''Brownian'' motion model [5]. Through the reduction of the FP equation to the Schrödinger equation, the latter model is already known to be connected to the Calogero Hamiltonian and thereby to various other models [6–8]. In this paper, we explore RM models with noninvariant distributions, and, following the same route as in the case of SGE, connect them to the Calogero Hamiltonian. This gives us a new technique to analyze the spectral behavior of the quantum operators of complex systems.

The connection between complex systems and the Calogero Hamiltonian seems to be wide ranging. The eigenvalue dynamics of Hermitian operators, for example, Hamiltonians of complex quantum systems, e.g., chaotic systems, disordered systems seem to have an intimate connection with the particle dynamics of the Calogero-Moser (CM) Hamiltonian. The latter describes the dynamics of particles interacting via pairwise inverse square interaction and is confined to move along a real line [1],

$$\hat{H}=-\sum_i \frac{\partial^2}{\partial\mu_i^2}+\frac{1}{4}\sum_{i<j}\frac{\beta(\beta-2)}{(\mu_i-\mu_j)^2}-\sum_i V(\mu_i). \qquad (1)$$

Here $\mu_i$ is the position of the $i$th particle and $V(\mu_i)$ is the confining potential. Similarly the level dynamics of the unitary operators, e.g., the time-evolution operator is connected to the Calogero-Sutherland (CS) Hamiltonian [9]

$$\hat{H}=-\sum_i \frac{\partial^2}{\partial\mu_i^2}+\frac{\beta(\beta-2)}{16}\sum_{i\neq j}\mathrm{cosec}^2\left(\frac{\mu_i-\mu_j}{2}\right)-\frac{\beta^2}{48}N(N^2-1), \qquad (2)$$

where particles are confined to move in a circle, thus mimicking the similar confinement of eigenvalues due to the unitary nature of the operator. The morphological transition caused by the interacting steps on a miscut crystal surface can also be modeled by the CS Hamiltonian [10]. Here the complexity is thermodynamic in nature. It is already well known that the parametric dispersion of the eigenvalues of the quantum system, with a nonintegrable classical limit, is described by a set of equations similar to the equations of motions of particles, in time, of the classical Calogero Hamiltonian [11]. This analogy extends also to the statistical properties in the two cases. The parametric evolution of the distribution $P_{N\beta}(\epsilon_1,\ldots,\epsilon_N;\tau)$ of the eigenvalues $\epsilon_i$ of a Hamiltonian $H=H_0+\tau H_1$ (of size $N$), with perturbation $H_1$ taken from a SGE corresponds to the time evolution of the distribution $P_{N\beta}(r_1,\ldots,r_N;t)$ of positions $r_i$'s of the particles and both the static as well as dynamical correlators of the eigenvalues turn out to be similar to those of the particles in the CM Hamiltonian [with $V(\mu)\propto\mu^2$ in Eq. (1)] [6–8]. Here $\beta$ refers to the generic symmetry class of the complex systems and therefore the transformation properties of the associated RM models (known as GOE, GUE, and GSE for $\beta=1,2$, and 4, where O, U, and S refer to the orthogonal, unitary, and symplectic nature of transformation and GE stands for Gaussian ensemble) [12,6]. In the limit $\tau\to\infty$, the eigenvalues attain an equilibrium distribution, known as Wigner-Dyson, which coincides with the probability distribution of $N$-particle coordinates $P_{N\beta}(r;t\to\infty)$ of the ground state of the CM Hamiltonian [3,8]. Similar analogies can also be made between the evolution of the eigenvalues of unitary operators $U=U_0e^{i\tau M}$, with $M$ taken from SGE, and the CS Hamiltonian [13]. This is equivalent to saying that the sta-

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tionary and nonstationary states of the CSM Hamiltonian correspond to the eigenvalue distribution of the systems subjected to random perturbations, strong ($\tau\to\infty$) and weak (finite $\tau$), respectively, and thereby to equilibrium and nonequilibrium distribution of SGE. In this paper we show a unique connection between the CM and RM model: a nonstationary state (finite $t$) of the CM Hamiltonian can also be mapped to the eigenvalue distribution of a generalized Gaussian ensemble (GGE); the correspondence is established by identifying a parameter $Y$ for GGE, equivalent to time $t$ for the CM Hamiltonian. This mapping can then be used to obtain the information about various spectral properties of GGEs.

In the recent past, RM ensembles have quite often been used to model the physical systems with complicated interactions [2,12]. The logic which could be given in support of the model is that the missing information about the detailed nature of the interactions can be mimicked by randomizing the associated generators of motion, that is, by taking their matrix representations as random matrices. However, as the specific details of the complexity of an operator should be reflected in the associated RM model, the distribution of the matrix elements can be of various types. For example, for a Hamiltonian with a chaotic classical limit (the least predictability of the long-term dynamics), the distribution can be chosen as Gaussian (the least information ensemble), with distribution parameters to be determined by the associated quantum dynamics. The corresponding RM model will thus belong to a generalized Gaussian ensemble with the matrix elements distribution given by $P(H)\propto e^{-f_1(H)f_2(H)}$ (with $f_1$ and $f_2$ arbitrary functions and $H$ as a typical matrix). The SGEs, with the matrix elements distribution given by $P(H)\propto e^{-\mathrm{Tr}\,H^2}$ are special cases of GGEs and many of their properties are already known. The various features of GGEs have, however, remained unknown so far. The purpose of this paper is to suggest a technique to fill in this gap in our information. As for SGE, the nature of matrix elements in GGE also depends on the exact symmetry conditions of the Hamiltonian and is again indicated by parameter $\beta$, with $\beta=1,2,4$ for a generic matrix element to be real, complex, or quaternion [12]. Here we discuss, in detail, the properties of the GG ensemble of complex Hermitian matrices ($\beta=2$); the GG ensemble of real-symmetric matrices ($\beta=1$) has been discussed elsewhere [14]. We also probe briefly the non-Gaussian ensembles that can serve as good models for complex systems with various conditions on the associated quantum dynamics.

We proceed as follows. Our technique is based on the statistical evolution of the eigenvalues of a GG ensemble with respect to a change in their distribution parameters. This requires prior information about the effect of a small change in the matrix element on eigenvalues and eigenvectors; the related study is given in Sec. I A. These results are then used to obtain, as described in Sec. I B, the distribution of eigenvalues $P(\mu,Y)$ of a matrix $H$ taken from a Gaussian ensemble, noninvariant under unitary transformation. The evolution of the eigenvalues is governed by a partial differential equation which, after certain parametric redefinitions, turns out to be formally the same as the FP equation for the Brownian motion of particles in Wigner-Dyson (WD) gas [12]. Section II contains the details of the reduction of the FP equation to the Schrödinger equation of the CM Hamiltonian and a mapping of their respective correlators. Section III deals with the application of our technique to some important physical processes, e.g., localization and a brief discussion of our technique applied to a few other important matrix ensembles is given in Sec. IV. Section V gives an alternative evolution equation for the eigenvalues. We conclude in Sec. VI, which is followed by the Appendixes containing the proofs of some of the results given in the main text of the paper.

## I. EIGENVALUE DISTRIBUTION OF GENERALIZED GAUSSIAN ENSEMBLES

### A. The change of eigenvalues and eigenfunctions

The eigenvalue equation of a complex Hermitian matrix $H$ is given by $HU=U\Lambda$ with $\Lambda$ as the matrix of eigenvalues $\lambda_n$ and $U$ as the eigenvector matrix, unitary in nature. As is obvious, a slight variation of the matrix elements of $H$ will, in general, lead to a variation of both the eigenvalues as well as the eigenvectors and associated rates of change can be obtained as follows.

As $\lambda_n=\Sigma_{i,j}U_{ni}H_{ij}U^*_{nj}$, the rate of change of $\lambda_n$ with respect to $H_{kl;s}$ (with $s$ referring to real, $s=1$, and imaginary, $s=2$, parts of $H_{kl}$) can be given

$$\frac{\partial\lambda_n}{\partial H_{kl;s}}=\frac{i^{s-1}}{g_{kl}}[U_{ln}U^*_{kn}-(-1)^sU^*_{ln}U_{kn}], \tag{3}$$

where $g_{kl}=1+\delta_{kl}$. This can further be used to obtain the following relations (Appendix A):

$$\sum_{k\leqslant l}\sum_{s=1}^{2}\frac{\partial\lambda_n}{\partial H_{kl;s}}H_{kl;s}=\sum_{k,l}H_{kl}U_{ln}U^*_{kn}=\lambda_n \tag{4}$$

and

$$\sum_{k\leqslant l}g_{kl}\sum_{s=1}^{2}\frac{\partial\lambda_n}{\partial H_{kl;s}}\frac{\partial\lambda_m}{\partial H_{kl;s}}=2\delta_{mn}. \tag{5}$$

For our analysis later, we also require the information about the second-order change of an eigenvalue with respect to a matrix element and, therefore, the rate of change of one of the eigenvector components with respect to $H_{kl}$. This is given as follows (Appendix B):

$$\frac{\partial U_{pn}}{\partial H_{kl;s}}=\frac{i^{s-1}}{g_{kl}}\sum_{m\neq n}\frac{1}{\lambda_n-\lambda_m}U_{pm}[U^*_{km}U_{ln}+(-1)^{s+1}U^*_{lm}U_{kn}], \tag{6}$$

and now by using Eqs. (3) and (6), one can show that (Appendix C)

$$\sum_{k\leqslant l}g_{kl}\sum_{s=1}^{2}\frac{\partial^2\lambda_n}{\partial H^2_{kl;s}}=4\sum_{m}\frac{1}{\lambda_n-\lambda_m}. \tag{7}$$

For the real-symmetric case, the corresponding relations can be obtained by using $U^+=U^T$ (as the eigenvector matrix is now orthogonal) in Eqs. (3)–(7) and taking $H_{ij;2}=0$ for all values of $i,j$ (see [8]).

### B. The evolution equation for the eigenvalues

Let us consider an ensemble of complex Hermitian matrices $H$, with matrix elements $H_{kl}=H_{kl;1}+iH_{kl;2}(1-\delta_{kl})$ distributed as Gaussians with arbitrary variances and mean values; the variances of real and imaginary parts of a single matrix element also need not be same. Thus we choose the distribution $\rho(H)$ of matrix $H$ to be following:

$$\rho(H,y,b)=C\exp\left(-\sum_{s=1}^{2}\sum_{k\leqslant l}\alpha_{kl;s}(H_{kl;s}-b_{kl;s})^2\right) \quad (8)$$

with $C=\Pi_{k\leqslant l}\Pi_{s=1}^{2}\sqrt{\alpha_{kl;s}/\pi}$ as the normalization constant, $y$ as the set of coefficients $y_{kl;s}=\alpha_{kl;s}g_{kl}=g_{kl}/2\langle H_{kl;s}^2\rangle$, and $b$ as the set of all $b_{kl;s}$. Note that such a choice leads to a nonrandom complex Hamiltonian ($H_{kl}=b_{kl;1}+ib_{kl;2}$) in the limit $\alpha_{kl;1},\alpha_{kl;2}\to\infty$ and, therefore, can model various real physical situations such as the switching of disorder in a nonrandom Hamiltonian, e.g., metal-insulator transitions.

Let $P(\mu,y,b)$ be the probability of finding eigenvalues $\lambda_i$ of $H$ between $\mu_i$ and $\mu_i+d\mu_i$ at a given $y$ and $b$,

$$P(\mu,y,b)=\int\prod_{i=1}^{N}\delta(\mu_i-\lambda_i)\rho(H,y,b)dH. \quad (9)$$

As the $\alpha$ dependence of $P$ in Eq. (9) enters only through $\rho(H)$ and $\partial\rho/\partial\alpha_{kl;s}=[(2\alpha_{kl;s})^{-1}-(H_{kl;s}-b_{kl;s})^2]\rho=(2\alpha_{kl;s})^{-1}[\rho+(H_{kl;s}-b_{kl;s})(\partial\rho/\partial H_{kl;s})]$ with $\partial\rho/\partial H_{kl;s}=-\partial\rho/\partial b_{kl;s}$, a derivative of $P$ with respect to $\alpha_{kl;s}$ can be written as follows:

$$\frac{\partial P}{\partial\alpha_{kl;s}}=\frac{P}{2\alpha_{kl;s}}+\frac{1}{2\alpha_{kl;s}}\int\prod_{i=1}^{N}\delta(\mu_i-\lambda_i)H_{kl;s}\frac{\partial\rho}{\partial H_{kl;s}}dH+\frac{1}{2\alpha_{kl;s}}\int\prod_{i=1}^{N}\delta(\mu_i-\lambda_i)b_{kl;s}\frac{\partial\rho}{\partial b_{kl;s}}dH. \quad (10)$$

The second integral in Eq. (10) is equal to $b_{kl;s}(\partial P/\partial b_{kl;s})$. The first integral can also be simplified by using integration by parts followed by the use of the equality $\partial\Pi_{i=1}^{N}\delta(\mu_i-\lambda_i)/\partial H_{kl;s}=-\Sigma_{n=1}^{N}[\partial\Pi_{i=1}^{N}\delta(\mu_i-\lambda_i)/\partial\mu_n](\partial\lambda_n/\partial H_{kl;s})$:

$$\int\prod_{i=1}^{N}\delta(\mu_i-\lambda_i)H_{kl;s}\frac{\partial\rho}{\partial H_{kl;s}}dH=-\int\frac{\partial\prod_{i=1}^{N}\delta(\mu_i-\lambda_i)}{\partial H_{kl;s}}H_{kl;s}\rho dH-\int\prod_{i=1}^{N}\delta(\mu_i-\lambda_i)\rho dH=I_{kl;s}-P, \quad (11)$$

where

$$I_{kl;s}=\sum_{n=1}^{N}\frac{\partial}{\partial\mu_n}\int\prod_{i=1}^{N}\delta(\mu_i-\lambda_i)\frac{\partial\lambda_n}{\partial H_{kl;s}}H_{kl;s}\rho dH. \quad (12)$$

Substitution of Eq. (11) in Eq. (10) then gives

$$2\alpha_{kl;s}\frac{\partial P}{\partial\alpha_{kl;s}}=I_{kl}+b_{kl;s}\frac{\partial P}{\partial b_{kl;s}}. \quad (13)$$

Our aim is to find a function $Y$ of the coefficients $\alpha_{kl;s}$'s and $b_{kl;s}$'s such that the evolution of $P(\mu,Y)$ in terms of $Y$ satisfies a FP equation similar to that of Dyson's Brownian motion model (Wigner-Dyson gas) [5,12]. For this purpose, we consider the sum $2\Sigma_{k\leqslant l}(\gamma-g_{kl}\alpha_{kl;s})\alpha_{kl;s}(\partial P/\partial\alpha_{kl;s})$, where $\gamma$ is an arbitrary parameter, and thereby obtain the following relation:

$$\sum_{s=1}^{2}\sum_{k\leqslant l}(\gamma-y_{kl;s})\left[2y_{kl;s}\frac{\partial P}{\partial y_{kl;s}}-b_{kl;s}\frac{\partial P}{\partial b_{kl;s}}\right]=\sum_{s=1}^{2}\left[\gamma\sum_{k\leqslant l}I_{kl;s}-\sum_{k\leqslant l}y_{kl;s}I_{kl;s}\right]. \quad (14)$$

As shown in Appendix D, the first term on the right-hand side of Eq. (14) can further be simplified:

$$\sum_{s=1}^{2}\sum_{k\leqslant l}I_{kl;s}=\sum_{n}\frac{\partial}{\partial\mu_n}(\mu_nP). \quad (15)$$

The second term can similarly be rewritten as follows (Appendix E):

$$\sum_{s}\sum_{k\leqslant l}y_{kl;s}I_{kl;s}=-\sum_{n}\frac{\partial}{\partial\mu_n}\left[\frac{\partial}{\partial\mu_n}+\sum_{m\neq n}\frac{\beta}{\mu_m-\mu_n}\right]P-\sum_{k\leqslant l}y_{kl;s}b_{kl;s}\frac{\partial P}{\partial b_{kl;s}}, \quad (16)$$

where $\beta=2$. Using both the equalities (15) and (16) in Eq. (14), we obtain the desired FP equation

$$\frac{\partial P}{\partial Y}=\gamma\sum_{n}\frac{\partial}{\partial\mu_n}(\mu_nP)+\sum_{n}\frac{\partial}{\partial\mu_n}\left[\frac{\partial}{\partial\mu_n}+\sum_{m\neq n}\frac{\beta}{\mu_m-\mu_n}\right]P. \quad (17)$$

Here the left-hand side of the above equation, summing over all $y_{kl;s}$ and $b_{kl;s}$, has been rewritten as $\partial P/\partial Y$ with $Y$ given by the condition that

$$\frac{\partial P}{\partial Y}=2\sum_{s}\sum_{k\leqslant l}y_{kl;s}(\gamma-y_{kl;s})\frac{\partial P}{\partial y_{kl;s}}-\gamma\sum_{s}\sum_{k\leqslant l}b_{kl;s}\frac{\partial P}{\partial b_{kl;s}}. \quad (18)$$

By using the orthogonality of eigenvectors and following the same steps, it can be proved for a real symmetric case also (now $\beta=1$) [14]. It is worth noting that Eq. (17) is the same as the evolution equation for the eigenvalues of Brownian ensembles. It is also similar to the one governing the transitions between any two universality classes of SGE caused by a random perturbation of strength $\tau$ (with $\tau\to Y$) [8,6].

### C. How to obtain the complexity parameter $Y$

The variable $Y$, a function of relative values of the coefficients $\alpha_{kl;s}$'s and $b_{kl;s}$'s, is a measure of the degree and

nature of the complexity of a system and can therefore be referred to as the ''complexity parameter.'' For the case discussed here [Eq. (18)], $Y$ can be obtained by the following method.

We define $M=2N^2$ variables $(Y_1,\ldots,Y_M)$ as the functions of all $y_{kl;s}$'s and $b_{kl;s}$'s such that the condition given by Eq. (18) (where $Y\equiv Y_1$) is satisfied. This is indeed possible by using the orthogonal (Jacobi) coordinate transformation between variables $\{Y_i\}_{i=1,\ldots,M}$ and $\{y_{kl;s},b_{kl;s}\}_{k\leqslant l;k,l=1,\ldots,N;s=1,2}$ defined by the following rule:

$$Y_i=\sum_{j=1}^{M} a_{ij}X_j \quad \text{for} \quad i=1\rightarrow M, \tag{19}$$

where $X_j\equiv\frac{1}{2}\ln(y_{kl;s}/|y_{kl;s}-\gamma|)+c_j$ for $j\leqslant N^2$ and $X_j\equiv-(1/\gamma)\ln|b_{kl;s}|+c_j$ for $j>N^2$, with $c_j$ as arbitrary constants of integration. Here coefficients $a_{ij}$ must satisfy the relation $\Sigma_{j=1}^{M}a_{ij}=\delta_{i1}$ which is a necessary condition for orthogonality but not sufficient to get the right form for $\partial/\partial Y$. With $D$ being the functional derivative of $Y_i$'s with respect to $X_j$'s, we also need the elements $D_{1j}^{-1}$ of its inverse to be unity. One way to achieve this is to set all adjuncts of the matrix elements $\partial Y_1/\partial X_j$ equal. Now by choosing $a_{1j}$ also equal, $a_{1j}=M^{-1}$, we are left with $M$ conditions for $a_{ij}$, $i\neq 1$, which can easily be fulfilled.

The form of $Y=\Sigma_j a_{1j}X_j$, fulfilling condition (18), can therefore be given as

$$Y=\frac{1}{2N^2}\sum_{k\leqslant l}\sum_{s=1}^{2}\left[\frac{1}{2}\ln\frac{y_{kl;s}}{|y_{kl;s}-\gamma|}-\frac{1}{\gamma}\ln|b_{kl;s}|\right]+C \tag{20}$$

with $C=M^{-1}\Sigma_j c_j$.

As is obvious, this method is applicable only for the case when the prefactor associated with a derivative of $P$ with respect to a variable $r$ in Eq. (18) depends only on $r$ ($r$ can be any one of the $y_{kl;s}$ or $b_{kl;s}$). Our studies on the ensembles more complicated than Eq. (8) show that the prefactors can also depend on a combination of various $r$ variables. This requires a more general method to obtain $Y$ which can also be used for the case discussed here (Appendix F).

### D. Determination of $P(\mu,Y)$

Equation (17) describes an evolution of the eigenvalues of GGE due to changing distribution parameters of the ensemble that can be solved, in principle, to obtain $P(\mu,Y)$ for arbitrarily chosen initial values of the parameters. If the ensemble corresponding to the initial set of parameters is referred as $H_0$, an integration over $H_0$ would lead to $P(\mu,Y)$, free of initial conditions. In fact, it can be shown that

$$P(\mu;Y)=(4\pi Y)^{-N^2/2}\int \exp\left[-\frac{1}{4Y}\mathrm{tr}(\mu-U^{+}\mu_0 U)^2\right] \times f(\mu_0)|\Delta(\mu_0)|^{\beta_0}d\mu_0 dU, \tag{21}$$

where $\mu_0$ is the set of eigenvalues of the initial matrix $H_0$, with $\beta_0$ given by its symmetry conditions, and $U$ is the integral over unitary (or orthogonal) space of matrices.

To show that Eq. (21) is indeed a solution of Eq. (17), we study a general case. Consider a partial differential equation for a function $F(A;t)$ defined in the matrix space of $N\times N$ Hermitian matrices $A$

$$\frac{\partial F}{\partial t}=[\nabla_A^2 F+\nabla\cdot(AF)], \tag{22}$$

where

$$\nabla_A^2=\sum_i\frac{\partial^2}{\partial A_{ii}^2}+\frac{1}{2}\sum_{i<j}\frac{\partial^2}{\partial A_{ij}^2},$$

$$\nabla\cdot(AF)=\sum_{i\leqslant j}\frac{\partial}{\partial A_{ij}}(A_{ij}F), \tag{23}$$

with the initial condition $F(A;0)=f(A)$. This equation is known to have a unique solution (see p. 174 of [12]),

$$F(A;t)=\int K(A,B,t)f(B)dB, \tag{24}$$

where

$$K(A,B,t)=(4\pi t)^{-N^2/2}\exp\left[-\frac{1}{4t}\mathrm{tr}(A-B)^2\right], \tag{25}$$

where $B$ is an $N\times N$ Hermitian matrix. Depending on the nature of both $A$ and $B$, we can choose a special class of eigenvector matrices $U_A$ and $U_B$ (for $A$ and $B$ real symmetric, complex Hermitian, or symplectic, $U_A$ and $U_B$ are orthogonal, unitary, and symplectic matrices, respectively) such that

$$A=U_A^s a U_A \quad \text{and} \quad B=U_B^s b U_B, \tag{26}$$

where $a=[a_i\delta_{ij}]$, $b=[b_i\delta_{ij}]$ are diagonal matrices with $a_i$ and $b_i$ as the eigenvalues of $A$ and $B$, respectively, and $U^s=U^{+}$ or $U^T$ or $U^R$, depending on whether $U$ is an eigenvector matrix for a complex Hermitian, real symmetric, or symplectic matrix [12]. Let $\beta_A$ and $\beta_B$ give the number of components of typical matrix elements in $A$ and $B$, respectively. Changing the variables from matrix elements to the $N$ eigenvalues and $\beta N(N-1)/2$ angle (i.e., eigenvector) parameters on which $U_B$ depends, we have

$$dB=|\Delta(b)|^{\beta_b}dbdU_B$$

with

$$db=\prod_{i=1}^{N}db_i \quad \text{and} \quad \Delta(b)=\prod_{i\leqslant j}(b_i-b_j). \tag{27}$$

The substitution of these relations in Eq. (24) gives us

$$F(A;t)=(2\pi t)^{-N^2/2}\int \exp\left[-\frac{1}{2t}\mathrm{tr}(a-U^s b U)^2\right] \times f(b,U_B)|\Delta(b)|^{\beta_b}dbdU_B, \tag{28}$$

where $U=U_B U_A^s$ and $U^s=U_A U_B^s$. Now if $f(b,U_B)$ is independent of $U_B$ then $F(A;t)$ is also independent of $U_A$. This helps us to rewrite Eq. (28) as follows:

$$F(a;t)=(4\pi t)^{-N^2/2}\int G(a,b,t)f(b)|\Delta(b)|^{\beta_b}db, \quad (29)$$

where

$$G(a,b,t)=\int \exp\left[-\frac{1}{4t}\mathrm{tr}(a-U^s b U)^2\right]dU.$$

Here the integral is over the group $U$ of orthogonal, unitary, and symplectic matrices, respectively. Further the Laplacian $\nabla_A^2$ can also be written in terms of eigenvalues and angle parameters of $A$ (see Appendix A.5 of [12]),

$$\nabla^2(A)=\frac{1}{|\Delta(a)|^{\beta_a}}\sum_i \frac{\partial}{\partial a_i}|\Delta(a)|^{\beta_a}\frac{\partial}{\partial a_i}+\nabla^2_{U_A}. \quad (30)$$

By the substitution of Eq. (30) in Eq. (22) and using the independence of $F(a;t)$ of $U_A$, one can rewrite Eq. (22) as follows:

$$\frac{\partial F(a;t)}{\partial t}=\frac{1}{|\Delta(a)|^{\beta_a}}\sum_i \frac{\partial}{\partial a_i}\left[|\Delta(a)|^{\beta_a}\frac{\partial F(a;t)}{\partial a_i}\right]+\sum_i \frac{\partial}{\partial a_i}(a_i F) \quad (31)$$

with $F(a;t)$ given by Eq. (29). Now by using the equality $\Sigma_i(\partial^2/\partial a_i^2)|\Delta(a)|^{\beta_a}=0$, Eq. (31) can be reduced in the following form:

$$\frac{\partial F}{\partial t}=\sum_i \frac{\partial}{\partial a_i}(a_i F)+\sum_i \frac{\partial}{\partial a_i}\left[\frac{\partial}{\partial a_i}+\sum_{j\neq i}\frac{\beta_a}{a_j-a_i}\right]F, \quad (32)$$

which is similar to Eq. (17) with $a_i\to\mu_i$, $t\to Y$, $\gamma=1$, and $F\to P$. The joint probability density $P$ can therefore be obtained by evaluating the integral (29). However, so far, the integration could be performed only for the unitary group of matrices [15,16,12].

### E. Steady state, level density, and correlations

The steady state of Eq. (17), $P(\mu,\infty)\equiv P_\infty=|\Delta(\mu)|^\beta e^{-(\gamma/2)\Sigma_k\mu_k^2}$, corresponds to $Y-Y_0\to\infty$ (with $Y_0$ as the complexity parameter of the initial ensemble) which can be achieved by two ways (for finite $Y_0$ values). The first is when almost all $y_{kl;1}\to\gamma$ and $y_{kl;2}\to\infty$ (for finite $b_{kl;1}$ and $b_{kl;2}$ values) which results in a GOE steady state. The second is when almost all $y_{kl;1}\to\gamma, y_{kl;2}\to\gamma$, resulting in a GUE. This indicates that, in the steady-state limit, the system tends to belong to one of the SGEs. Equation (17) can, therefore, describe a transition from a given initial ensemble (with $Y=Y_0$) to either GOE or GUE with $Y-Y_0$ as the transition parameter. The nonequilibrium states of this transition, given by nonzero finite values of $Y-Y_0$, are various Gaussian ensembles corresponding to varying values of the coefficients $y_{kl;s}$ and $b_{kl;s}$. For example, the choice of the initial ensemble as GOE (almost all $y_{kl;1}=\gamma, y_{kl;2}\to\infty$ initially) and a decrease of $y_{kl;2}$ (from $\infty\to\gamma$ while keeping $y_{kl;1}$ fixed) leads to the GOE→GUE transition with intermediate ensembles as those of complex Hermitian matrices. Similarly Poisson → GUE transition can be brought about by a choice of the initial ensemble as Poisson (almost all $y_{kl;1}, y_{kl;2}\to\infty$ for $k\neq l$, $y_{kk;1}=\gamma$, $y_{kk;2}=\gamma$, and $b_{kl;s}=0$ for all $k,l,s$ values) and by varying both $y_{kl;1}$ and $y_{kl;2}$ up to $\gamma$. As is clear from above, $\gamma$ fixes the variance of the final ensemble and an arbitrariness in $\gamma$ leaves the latter arbitrary. This, however, does not affect the statistical properties of the intermediate ensembles.

The equation (21) for $P(\mu,Y)$ can be used to obtain the $n$th-order density correlator $R_n(\mu_1,\ldots\mu_n;Y)$, defined by $R_n=[N!/(N-n)!]\int P(\mu,Y)d\mu_{n+1}\cdots d\mu_N$. ($R_n$ can also be expressed in the form $\langle\nu(\mu_1,Y)\cdots\nu(\mu_n,Y)\rangle$ with $\nu(\mu,Y)=N^{-1}\Sigma_i\delta(\mu-\mu_i)$ as the density of eigenvalues and $\langle\cdots\rangle$ implying the ensemble average). Here note that the analogy of Eq. (17) to that of Dyson's Brownian ensembles (BE) implies the same form of $P$ for both the cases and therefore $R_n$. A lot of information is already available about level density and various correlations for Brownian ensembles; it can directly be used for the ensemble described by Eq. (8). Thus, as for BE, a direct integration of the FP equation (17) leads to the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchic relations among the unfolded local correlators

$$R_n(r_1,\ldots,r_n;\Lambda)=\lim_{N\to\infty}\frac{R_n(\mu_1,\ldots,\mu_n;Y)}{R_1(\mu_1;Y)\cdots R_1(\mu_n;Y)}$$

with $r=\int^r R_1(\mu;Y)d\mu$ and $\Lambda=(Y-Y_0)/D^2$ ($D=R_1^{-1}$; the mean level spacing) [13],

$$\frac{\partial R_n}{\partial\Lambda}=\sum_j\frac{\partial^2 R_n}{\partial r_j^2}-\beta\sum_{j\neq k}\frac{\partial}{\partial r_j}\left(\frac{R_n}{r_j-r_k}\right)-\beta\sum_j\frac{\partial}{\partial r_j}\int_{-\infty}^{\infty}\frac{R_{n+1}}{r_j-r_k} \quad (33)$$

(here, for simplification, $\gamma$ is chosen to be unity). As can be seen from the above equation, the transition for $R_n$ occurs on the scales determined by $Y\approx D^2$ and a smooth transition can be brought only in terms of the parameter $\Lambda$, obtained by rescaling $Y$ by $D^2$. On the other hand, for $R_1$, the corresponding scale is given by $Y\approx ND^2$. This implies, therefore, during the transition in $R_n$, the density $R_1$ remains nearly unchanged; this fact is very helpful in unfolding the correlators $R_n$. For $n=1$ and in large $N$ limit, the above equation reduces to the Dyson-Pastur equation [13] for the level density $\langle\nu(\mu_1,Y)\rangle\equiv N^{-1}R_1$,

$$\frac{\partial\langle\nu(\mu)\rangle}{\partial Y}=-\beta\frac{\partial}{\partial\mu}\left(\sum_m \mathrm{P}\int d\mu'\frac{\langle\nu(\mu')\rangle}{\mu-\mu'}\right)\langle\nu(\mu)\rangle, \quad (34)$$

which results in a semicircular form for $\nu$;

$$\nu(r)=(2/\pi\eta^2)(\eta^2-r^2)^{1/2}$$

with $\eta^2=4N(1+Y^2)$ [17]. The application of the supersymmetry (SUSY) technique [18] to ensemble (8) gave a similar result (also see Sec. 4.3 of [12]).

## II. CONNECTION TO THE CALOGERO HAMILTONIAN

A similarity transformation followed by a Wick rotation converts the FP equation into a self-adjoint form [8]. This can be seen as follows. The FP equation, in general, can be expressed in a form

$$\frac{\partial|P_Y\rangle}{\partial Y}=-P|P_Y\rangle, \tag{35}$$

where $P$ is a FP operator with non-negative eigenvalues. Here $|P_Y\rangle$ is a general state of operator $P$ at ''time'' $Y$ and its projection in eigenvalue space can be obtained by the usual operation $P(\mu,Y)\equiv\langle\mu|P_Y\rangle$ (with $\mu$ as set of the eigenvalues). Let $P(\mu,Y_0)\equiv\langle\mu|P_0\rangle$ be the equilibrium probability. One can further define a vector $\langle 0|\equiv\int d\mu\langle\mu|\ldots$ satisfying $\langle 0|P=0$ thus implying the conservation of probability in ''time'' $Y$ in this state (the ground state). The FP operator can now be Hermiticized through a similarity transformation $S^{-1}PS=H$, where $S$ is the Hermitian and invertible operator depending only on the eigenvalues. Thus the ground-state condition must be given by $HS|0\rangle=0$ (as $P^+|0\rangle=0$). Let the effect of similarity transformation on the state $|P_Y\rangle$ and $|P_0\rangle$ is expressed by $|\psi\rangle=S^{-1}|P_Y\rangle$ and $|\psi_0\rangle=S^{-1}|P_0\rangle$, respectively. The similarity transformation of Eq. (35) will then give the desired form $\partial|\psi\rangle/\partial Y=-H|\psi\rangle$; the ground state $|\psi_0\rangle$ must now satisfy the condition $H|\psi_0\rangle=0$. The comparison of the two different forms of the ground-state condition gives $|\psi_0\rangle=S|0\rangle$ and therefore $|P_0\rangle=S^2|0\rangle$.

In the case of the FP equation (17), $H$ turns out to be a CM Hamiltonian [Eq. (1) with $r_i\rightarrow\mu_i$] and has well-defined eigenstates and eigenvalues [1,19]. As is well known, the particles in the CM system undergo an integrable dynamics, thus implying a similar motion for the eigenvalues, too. Here $H$ is a generic member of GGE; this result is valid for all systems with interactions complicated enough to be modeled by GGE.

The ''state'' $\psi$ or $P(\mu,Y|H_0)$ can be expressed as a sum over the eigenvalues and eigenfunctions which on integration over the initial ensemble $H_0$ leads to the joint probability distribution $P(\mu,Y)$ and thereby static (at a single parameter value) density correlations $R_n$. The above correspondence can also be used to map the multiparametric correlations to multitime correlations of the of CM Hamiltonian. For example, the parametric correlation $\langle Q_a(Y)Q_b(0)\rangle$, for a classical variable $Q(Y)$ with $[Q,S]=0$ can be mapped to the corresponding ground-state correlation of the CM Hamiltonain $\langle\psi_0|Q_a(Y)e^{-YH}Q_b(0)|\psi_0\rangle$. This follows because

$$\begin{aligned}\langle Q_a(Y)Q_b(0)\rangle&=\int Q_aQ_bP(\mu,Y)d\mu\\&=\int\langle\mu|Q_aQ_b|P_Y\rangle d\mu.\end{aligned} \tag{36}$$

Now as the evolution of $|P_Y\rangle$ with respect to $Y$ is given by $|P_Y\rangle=Se^{-YH}S^{-1}|P_0\rangle$, one has

$$\begin{aligned}\langle Q_a(Y)Q_b(0)\rangle&=\langle 0|Q_aSe^{-YH}S^{-1}Q_b|P_0\rangle\\&=\langle\psi_0|Q_ae^{-YH}Q_b|\psi_0\rangle.\end{aligned} \tag{37}$$

## III. APPLICATION TO PHYSICAL PROBLEMS

The given ensemble (8), referred to here as ''G,'' is represented by a point $Y$ in the parametric space, consisting of distribution parameters and various transition curves may pass through this point. The question therefore arises as to which curve should be chosen for the studies of the properties of $G$? The answer is the one which does not leave any arbitrariness behind and if there is more than one such curve, each one of them should give the same answer for various fluctuation measures of $G$. These criteria for the right choice are based on the symmetry properties of ensemble $G$, that is, the nature of all $\alpha_{kl}$ and $b_{kl}$ with endpoints (the final and initial ensemble, referred to here as ''$F$'' and ''$O$,'' respectively) chosen in such a way that the values corresponding to $G$ occur during the variation of distribution parameters from one end to the other. Furthermore, the chosen transition should preferably be the one whose properties are already known and can therefore tell us about $G$. For many GGE described by Eq. (8), the above criteria is satisfied by choosing $F$ as a SGE with variance $\langle F_{ii}^2\rangle=2\langle F_{ij}^2\rangle=\gamma^{-1}$, $\gamma\leqslant\min\{y_{kl;s}[G]\}$, $k,l=1,2,\ldots,N$, $s=1,2$, and $O$ as an ensemble with each $\alpha_{kl}[O]$ given by the maximum value taken by the functional form of the corresponding $\alpha_{kl}[G]$. However, as explained in the following examples, $O$ can also be chosen as some other ensemble. For example, If $G$ is an ensemble of real-symmetric matrices $H$ represented by $\rho(H)\propto\exp[-\Sigma_{k\leqslant l}\alpha_{kl}H_{kl}^2]$ with finite but different values for all $\alpha_{kl}$, the Poisson$\rightarrow$GOE curve is more suitable for its study rather than GOE $\rightarrow$GUE. Here the GOE ensemble is described by $\langle F_{ii}^2\rangle=2\langle F_{ij}^2\rangle=\gamma^{-1}$ with $\gamma$ as the minimum value among given $y_{kl}[G]$s. However, if various $\alpha_{kl}$ in the above example can also take complex values, the ensemble can now be chosen on any one of the curves, namely, Poisson $\rightarrow$ GUE or GOE $\rightarrow$ GUE. Here now GUE can be chosen as $\langle F_{ii}^2\rangle=2\langle F_{ij;1}^2\rangle=2\langle F_{ij;2}^2\rangle=\gamma^{-1}$. The GOE for the second curve can be chosen as the one with $\langle O_{ii}^2\rangle=2\langle O_{ij;1}^2\rangle=q^{-1}$ and $\langle O_{ij;2}^2\rangle=0$ with $q=\max\{y_{ij;1}[G]\}$. Similarly, for the Poisson $\rightarrow$ GUE curve, the initial ensemble may be taken as one with $\langle O_{ii}^2\rangle=\gamma^{-1}$ and $\langle O_{ij;1}^2\rangle=\langle O_{ij;2}^2\rangle=0$ for $i\neq j$. The reason for the choice of the two transitions is due to the availability of the results for their two-point correlation $R_2$ [13]:

For Poisson $\rightarrow$ GUE

$$\begin{aligned}R_2(r;\Lambda)-R_2(r;\infty)=\frac{4}{\pi}\int_0^\infty dx\int_{-1}^1 dz\cos(2\pi rx)\\\times\exp[-8\pi^2\Lambda x(1+x+2z\sqrt{x})]\\\times\left(\frac{\sqrt{(1-z^2)}(1+2z\sqrt{x})}{1+x+2z\sqrt{x}}\right)\end{aligned} \tag{38}$$

and for GOE $\rightarrow$ GUE

$$R_2(r;\Lambda)-R_2(r;\infty)$$
$$=-\frac{1}{\pi^2}\int_0^{\pi}dx\int_{\pi}^{\infty}dz\,x\sin(rx)\exp[2\Lambda(x^2-y^2)]\frac{\sin(yr)}{y}, \tag{39}$$

where $R_2(r,\infty)=1-[\sin^2(\pi r)/\pi^2 r^2]$ (the GUE limit).

It is obvious therefore that if $\Lambda_1$ and $\Lambda_2$ are the parameter values for the ensemble $G$ on Poisson $\rightarrow$ GUE and GOE $\rightarrow$ GUE curves, respectively, one should have $R_{2,P\rightarrow U}(r;\Lambda=\Lambda_1)=R_{2,O\rightarrow U}(r;\Lambda=\Lambda_2)$. This would require an intersection of two curves in the $R_2-\Lambda$ space which, however, is possible. This is because the GOE can occur as an intermediate point in the Poisson $\rightarrow$ GUE transition. The GOE $\rightarrow$ GUE curve can also appear as a part of the Poisson $\rightarrow$ GUE curve; thus the choice of two different initial ensembles here corresponds only to two different origins of dynamics on the same curve.

The parameter $\gamma$, which determines $Y$ as well as the variances of $F$, enters into the calculation at a step given by Eq. (14) and can be chosen arbitrarily. As suggested by Eq. (17), the choice of different $\gamma$ values corresponds to different $Y$ values as well as the transition curves with endpoints of the same nature but different variances; this, however, would not imply different properties for the ensemble G (Appendix G). Similarly, although the FP equation is independent of the choice of the initial ensemble, the latter is required for the determination of the correlations of $G$. The possibility of an arbitrary choice of $O$ may seem to imply a certain arbitrariness left in the correlation of $G$. However, the choice of two different initial ensembles corresponds only to the two different origins of the dynamics approaching the same point in the parametric space. This will be clarified by the examples given below.

### A. Anderson transition

Using the above method, the transition parameter for a metal-insulator transition as a result of increasing disorder can exactly be calculated. To see this, let us consider the case of a $d$-dimensional disordered lattice, of size $L$, in the tight-binding approximation. Here, in the configuration space representation of the Hamiltonian, an $N\times N$ matrix of size $N=L^d$, the diagonal matrix elements will be site energies $\epsilon_i$. The hopping is generally assumed to connect only the $z$ nearest neighbors with amplitude $t$ so that the electron kinetic energy spread or bandwidth is $zt$. This therefore results in a sparse form of the matrix $H$. We first consider the case of the $L\rightarrow D$ transition brought about by decreasing diagonal disorder only. In this case, site energies $\epsilon_i$ are taken to be independent random variables with the probability density $p(\epsilon_i)$. In the Anderson model [20] of the metal-insulator (MI) transition, $p(\epsilon)$ was taken to be a constant $W^{-1}$ between $-W/2$ to $W/2$. Various physical arguments and approximations used in this case led to the conclusion that here all the states are localized for $W>4Kt\ln(W/2t)$ with $K$ as a function of $z$ and $d$.

However, as is well known now, the MI transition does not depend on the nature of $p(\epsilon)$ and the latter can also be chosen as Gaussian; the type of $p(\epsilon)$ affects only the critical point of the transition. The $\rho(H)$, for any intermediate state of MI transition brought about by diagonal disorder, can therefore be chosen as in Eq. (8) with $\alpha_{kl}\rightarrow\infty$, $b_{kl}=-t$ for $k\neq l$, $\alpha_{kk}=\alpha$, and $b_{kk}=0$ for all $k$ which results in

$$Y=\frac{1}{2N^2}\left[\frac{N}{2}\ln\frac{2\alpha}{|2\alpha-\gamma|}-\gamma^{-1}K\ln t\right]+C.$$

Here $K$ is total number of the sites connected and depends on the dimensionality $d$ of the system. The system can initially be considered in an insulator regime where all the eigenvectors become localized on individual sites of the lattice (strong disorder limit). This results in a diagonal form of the matrix $H$ with the eigenvalues independent from each other. The insulator limit can therefore be modeled by ensemble (8) with $\alpha_{kl}\rightarrow\infty$ for $k\neq l$, $\alpha_{kk}=\alpha_0$ (for all $k$ values) and $b_{kl}\rightarrow 0$ (for all $k,l$), giving $Y_0=(1/4N)\ln(2\alpha_0/|2\alpha_0-\gamma|)+C$ (as $K=0$ in the insulator regime). The decrease of the diagonal disorder, that is, an increase of $\alpha_{kk}$ from $\alpha_0$ to some finite values (while $\alpha_{kl}$, $k\neq l$, remains infinite throughout the transition) will ultimately lead to a metal regime with fully delocalized wave functions. The eigenvalue distribution of $H$ in the regime can be well modeled by the SGE; let it be described by $\alpha_M(>\alpha_0)$. Thus for the study of the transition in this case we should choose $\gamma=2\alpha_M$. The transition parameter can now be given as follows, with the mean level spacing $D\propto 1/\sqrt{N}$:

$$\Lambda=\frac{Y-Y_0}{D^2}=\frac{1}{4}\left[\ln\frac{\alpha|\alpha_0-\alpha_M|}{\alpha_0|\alpha-\alpha_M|}-\frac{K}{N\alpha_M}\ln t\right]. \tag{40}$$

As is obvious from the above, the transition is governed by relative values of the disorder and the hopping. Here $\Lambda\rightarrow 0$ leads a to fully localized regime that corresponds to the following condition on $\alpha$ and $t$:

$$\ln\frac{\alpha}{\alpha_0}+\frac{\alpha-\alpha_0}{\alpha_M}=\frac{K}{N\alpha_M}\ln t. \tag{41}$$

Equation (40) gives, therefore, the condition for the critical region or mobility edge ($K/N\rightarrow$finite as $N\rightarrow\infty$). As $(|\alpha-\alpha_0|/\alpha_M)\ll 1$ even for large $\alpha$ values, the condition is always satisfied if $K/N\alpha\rightarrow 0$. This explains the localization of all the states in infinitely long wires (or strictly 1D systems where $K\ll N$) even for very weak disorder. With increasing dimensionality $d$, connectivity $K$ of the lattice and thereby the possibility of $|\Lambda|\gg 0$ and the delocalized states increases. The $\Lambda$ can similarly be calculated when off-diagonal disorder is also present.

### B. 1D, quasi-1D, and periodic 1D disordered and chaotic systems

In 1D geometry of a solid state system, e.g., a chain of $N$ interacting sites, in tight-binding approximation, the long-range random hopping leads to a banded structure of the matrix, known as the random banded matrix (RBM) [21,22]. Here the effectively nonzero, randomly distributed, matrix elements are confined within a band with its width governed by the range of hopping. The 1D periodic geometry, e.g., a chain of interacting sites joined into a ring leads to a periodic RBM in which all nonzero matrix elements belong to three regions: a band along the main diagonal, the upper right

corner and the lower left one [21]. A real disordered wire has a finite cross section (referred to as quasi-1D geometry) and therefore allows for propagating modes with different transverse quantization numbers frequently referred to as transverse channels. This case can again be modeled by RBMs with the bandwidth given by the number of transverse channels [23,22]. In the case of dynamical systems also exhibiting strong chaos in the classical limit, the generic structure of the Hamiltonian matrix in some basis is banded and matrix elements can be assumed to be pseudorandom [24]. For example, the Hamiltonian of a quantum kicked rotor turns out to be a RBM in momentum basis [21].

In all these cases, the nature of the disorder or associated randomness decides the nature of the distribution of the matrix elements. The physical properties of such systems can therefore be analyzed by studying the distribution of the eigenvalues of associated RBMs. The most studied type of RBM is that with the zero mean value of all matrix elements and variance given by $\langle H_{nm}^2\rangle = v^2 a(|n-m|/b)$, where $a(r)$ is some function satisfying the condition $\lim_{r\to\infty} a(r)=0$ and determines the shape of the band [22]. For a large but finite size of the matrix $N \gg b \gg 1$, its statistical properties were shown (by the SUSY method) to be determined by the scaling parameter $b^2/N$ with the transition parameter scaling as $Nf(b^2/N)$ [21].

The transition parameter for the RBM can also be calculated by our method. Let us first consider the simplest case, i.e., the Rosenzweig-Porter (RP) model where all the off-diagonal matrix elements are distributed with the same variance which is different from the diagonal ones. Let us take $\alpha_{ij;i\neq j}[G]=2(1+\mu)$ and $\alpha_{ii}[G]=1$ with $\mu \geqslant 0$; thus $\min\{y_{ij}[G]\}=2$ and we can choose $\gamma=2$. This GGE can therefore be mapped to a Brownian ensemble, with

$$Y-Y_0 = -\frac{N-1}{4N}\ln\left|1-\frac{1}{1+\mu}\right| \approx \frac{1}{4\mu} \quad \text{for} \quad \mu>1,$$

appearing in a Poisson $\rightarrow$ GOE transition where the initial matrix elements distribution is given by $P(H_0)\propto e^{-\Sigma_i H_{ii}^2}$ and the final, stationary state, obtained for large $\Lambda$-values, is $P(H)\propto e^{-(\gamma/2)\mathrm{Tr}\,H^2}$. Now as $R_1 \approx \sqrt{N}$ [6,18], the $D^2 \approx 1/N$, and therefore $\Lambda \approx N/4\mu$ which implies that the GGE will have an eigenvalue statistics very different from that of Poisson or GOE only if $\mu \approx cN$ ($c$ is a finite constant). For $\mu > cN$, $\Lambda \to 0$ and for $\mu < cN$, $\Lambda \to \infty$ for $N\to\infty$, and thus the GGE behaves like a Poisson ensemble in the first case and like a GOE in the second case; this result is in agreement with the one obtained, in [25], by using the NLSM technique. (Note that in Ref. [25], $D$ is taken as $D\propto 1/N$, which gives $\Lambda \approx N^2/2\mu$ and therefore the GOE and Poisson ensemble result for $\mu < cN^2$ and $\mu > cN^2$, respectively.)

Consider the ensemble with the exponential decay of the variances away from the diagonal i.e., $\alpha_{kl}=e^{|k-l|/b}, k\leqslant l, 1 \ll b \ll N$. Thus, again $\gamma=2$ and the final ensemble is a SGE with $P(H)=e^{-(\gamma/2)\mathrm{Tr}\,H^2}$ and therefore $Y = -(1/4N^2)\Sigma_{i\leqslant j=1}^{N}\ln|1-\gamma y_{ij}^{-1}|+C$. Here the initial ensemble is that of the diagonal matrices with a Poisson distribution of the eigenvalues which corresponds to $y_{ii}[O]=2$ and $y_{ij;i\neq j}[O]\to\infty$ [this being the maximum value of $y_{kl}[G]$] giving $Y_0 = -(1/4N^2)\Sigma_{i=1}^{N}\ln|1-\gamma y_{ii}^{-1}[O]|+C$. Thus $Y-Y_0 = -(1/4N^2)\Sigma_{r=1}^{N}(N-r)\ln|1-2e^{-r/b}| \approx b/N$. As $R_1 \approx \sqrt{N}$, the transition parameter for the infinite system ($N\to\infty$) turns out to be $\Lambda = Y/D^2 \approx b$ (see [14]) which reconfirms that, in infinite systems, the transition is governed only by the bandwidth $b$ [22,21].

Another case of importance is the ensemble with power-law decay of variances $H_{ij}=G_{ij}a(|i-j|)$ with $G$ a typical member of SGE ($\langle G_{ii}^2\rangle = 2\langle G_{ij}^2\rangle = v^2$) and $a(r)=1$ and $(b/r)^\sigma$ for $r\leqslant b$ and $r>b$ ($b\gg 1$), respectively (known as the PRBM model where P stands for power) [26]. This corresponds to $y_{ij}=[1/v^2a^2(|i-j|)]$ and therefore $\gamma=\min\{y_{ij}\}=1/v^2$. Again, as for the exponential case, the choice of the initial and final ensembles remains the same. Now as $y_{ij} \equiv y_r = \gamma(r/b)^{2\sigma}$ (with $r=|i-j|$), we get

$$\begin{aligned}\Lambda &= D^{-2}(Y-Y_0)\\ &= -\frac{1}{4N}\sum_{r=b+1}^{N}(N-r)\ln\left[1-\left(\frac{b}{r}\right)^{2\sigma}\right]\\ &\approx \frac{N}{4}\sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{b}{N}\right)^{2j\sigma}\int_{b/N}^{1}dx(1-x)x^{-2j\sigma} \qquad (42)\\ &= \frac{N}{4}\sum_{j=1}^{\infty}\frac{1}{j}\left[\frac{1}{2(1-2j\sigma)(1-j\sigma)}\left(\frac{b}{N}\right)^{2j\sigma}-\frac{1}{(1-2j\sigma)}\frac{b}{N}\right.\\ &\quad \left.+\frac{1}{2(1-j\sigma)}\left(\frac{b}{N}\right)^{2}\right]. \qquad (43)\end{aligned}$$

Thus, for large $N$ values and $\sigma<1/2$, $\Lambda(\propto N^{1-2\sigma})$ is sufficiently large and the eigenvalue statistics approaches the SGE limit. At $\sigma=1/2$, the statistics is governed by the parameter $b^2/N$ instead of $N$ only. For $\sigma=1$, the nonzero, finite $\Lambda$ value ($\Lambda\propto b$ even when $N\to\infty$) leads to an eigenvalue statistics intermediate between that of SGE or Poisson. For $\sigma>3/2$ with $N\to\infty$, $\Lambda\to 0$; therefore, the eigenvalue statistics approaches the Poisson limit, $\Lambda$ being very small. All these results are in agreement with those obtained in [26] by the SUSY technique.

Another type of RBMs often encountered in atomic and nuclear systems are those with the nonzero mean value of all matrix elements and with variance given by $\langle H_{nm}^2\rangle = v^2 a(|n-m|/b)$; the transition parameter for them can also be obtained as for the above cases [27–29,43,24,30].

### C. Quantum Hall case

A quantum Hall system without disorder has all the states degenerate within a single Landau level. The introduction of disorder leads to a broadening of the levels (also termed as diagonal disorder) as well as random hopping between them (off-diagonal disorder) and a competition between the two leads to a $L\to D$ transition. Note this is different from the Anderson model where the $L\to D$ transition is caused by the competition between diagonal disorder and nonrandom hopping (bandwidth) [20]. The $N\times N$ Hamiltonian matrix in the presence of disorder therefore belongs to an ensemble far more complicated than Eq. (8), known as the random Landau matrix, as now various matrix elements are no longer independently distributed:

$$\rho(H,y,b)=C\exp\left[-\sum_{s=1}^{2}\sum_{k,l;k\leqslant l}H_{kl;s}\left(\alpha_{kl;s}H_{kl;s}-\sum_{i,j;i\leqslant j}{}'\,b_{ijkl;s}H_{ij;s}\right)\right]$$

with $C$ as the normalization constant and $y$ and $b$ as the sets of inverse of variances $y_{kl;s}=\alpha_{kl;s}g_{kl}$ and coefficients $b_{ijkl;s}$, respectively, with $g_{kl}=1+\delta_{kl}$. Here $\Sigma'_{i,j}b_{ijkl;s}$ will imply that the summation is over all possible pairs of indices $\{i,j\}$ such that the pair $\{i,j\}\neq\{k,l\}$ or $\{l,k\}$ [31]. In this case, too, one can show that the eigenvalue distribution $P$ satisfies Eq. (17) but the condition for the determination of $Y$ is no longer given by Eq. (18); the details will be presented elsewhere.

### D. Critical ensemble and multifractality of eigenvectors

Recent studies of some metal-insulator transitions revealed that the energy level statistics in the critical region are universal and different from both Wigner-Dyson as well as Poisson statistics. The eigenfunctions associated with the critical statistics show multifractal characteristics [32–34]. The level number varaince $\Sigma^2(N)$ is believed to be an important indicator of this critical behavior with its asymptotic linearity in the mean number of levels $\overline{N}$ [35]; $\Sigma^2(\overline{N})=\langle(\delta N)^2\rangle=\chi\overline{N},\chi<1$. The critical statistics, therefore, governs the spectral fluctuations that are weaker than for the Poisson statistics $[\Sigma^2(\overline{N})=\overline{N}]$ but much stronger than for the Wigner-Dyson statistics, $[\Sigma^2(\overline{N})=\ln\overline{N}]$. Later on remarkable similarities were found between the spectral statistics of a number of dynamical systems, e.g., pseudointegrable billiards and the critical statistics near the mobility edge [36]. However, since such a critical region was inaccessible either perturbatively or semiclassically, a different tool was required to probe into it. This led to the suggestion of a RM modeling of this region [34]. The $N\times N$ matrices in this model are Hermitian and matrix elements are Gaussian distributed with zero mean and the variance given by

$$\langle(H_{ij})^2\rangle=\left[1+\left(\frac{|i-j|}{B}\right)^{2\sigma}\right]^{-1}. \tag{44}$$

Using the SUSY technique, it has been shown [26] that for large $B$ values ($B\gg1$), this ensemble behaves like a SGE for $\sigma<1$ and as a Poisson for $\sigma>1$. The case $\sigma=1$ is believed to be of special relevance as it supports critical statistics and multifractal eigenstates; the application of the SUSY technique gives $R_2(r)\approx1-(1/16B^2)[\sin^2(\pi r)/\sinh^2(\pi r/4B)]$ and $\Sigma^2(N)\approx\chi N$ [37,38,34].

The existence of the ensembles with critical statistics is indicated by our technique too. The $N$ dependence of the transition parameter $\Lambda$, entering through $Y$ and the mean level-spacing $D$, causes the transition to reach the equilibrium in limit $N\to\infty$ for finite, nonzero $Y$ values. In some cases, however, the $N$ dependence of $Y$ may be such that it balances the one due to $D$, thus resulting in an $N$-independent $\Lambda$ (as shown in Secs. III A and III B) and therefore critical statistics. As can be seen from Eq. (20), $\Lambda$ for the ensemble, given by Eq. (44), is also $N$ independent for $\sigma=1$; here the ensemble appears as an intermediate point between the Poisson → GUE transition with

$$Y-Y_0=\frac{1}{4N^2}\sum_{r=1}^{N}(N-r)\ln\left[1+\left(\frac{b}{r}\right)^{2\sigma}\right]$$

and $\Lambda$ behaves as in the case of the PRBM model discussed above, showing criticality for $\sigma=1$. The correlation $R_2$ for the ensemble (44) can therefore be given by Eq. (38), which for large $\Lambda$ values can be approximated as follows [39,40]:

$$R_2(r,\Lambda)=1+\frac{\Lambda}{\pi^2\Lambda^2+r^2}+\frac{1}{2\pi^2r^2}[\cos(2\pi r)e^{-2(r^2/\Lambda)}-1] \tag{45}$$

$$=1+\frac{1}{\pi^2\Lambda}+\frac{1}{2\pi^2r^2}[e^{-2(r^2/\Lambda)}-2e^{-2(r^2/\Lambda)}\sin^2(\pi r)-1]\quad(r\ll\sqrt{\Lambda}) \tag{46}$$

$$\approx1+\frac{1}{\pi^2\Lambda}-\frac{\sin^2(\pi r)}{\pi^2r^2e^{2r^2/\Lambda}}$$

$$\approx1-\frac{6}{\pi^2\Lambda}\frac{\sin^2(\pi r)}{\sinh^2(r\sqrt{6}/\Lambda)}, \tag{47}$$

which is similar to the result given by the SUSY technique. However, for $\Lambda\gg r\gg\sqrt{\Lambda}$, our method gives $1-R_2(r,\Lambda)=[-\Lambda/(\pi^2n^2+r^2)]+[1/(2n^2r^2)]$ while the SUSY technique gives $1-R_2$ as an exponentially decaying function.

As is obvious from Eq. (47), $R_2$ approaches the GUE limit as $\Lambda\to\infty$ but, for finite $\Lambda$ values, it is very different from both Poisson as well as GUE. This indicates that the ensembles with distribution parameters giving rise to a finite $\Lambda$ do not reach stationarity even for the infinite size of their matrices, and, their properties being very different from those of the equilibrium ensembles, can be referred to as ''critical.'' However, in our technique, as shown in previous sections, the difference between various $GG$ ensembles (within the same stationarity limits) manifests itself only in different $\Lambda$ values, leaving the functional form of various statistical measures unaffected. Thus the RP model as well as ensemble (44), both being GGEs and lying on the Poisson → GUE curve, would follow similar formulations for various statistical measures; For example, $R_2$ for both of them is given by Eq. (47) although with different formulas for $\Lambda$ and both can show the critical behavior. However, a contradiction arises when one considers the number variance statistics $\Sigma^2(r)$ which can be expressed in terms of $R_2(r)$ [12],

$$\Sigma^2(r;\Lambda)=r-2\int_0^r(r-s)[1-R_2(s)]ds \tag{48}$$

and therefore should show a similar behavior, as a function of $\Lambda$, for both [the RP model and ensemble (44)]. But a detailed study of the RP model by the SUSY technique [39] suggests that although it shows critical statistics for $\mu=cN$, it cannot support the linear nature of $\Sigma^2=\chi r$ with $\chi<1$. As

claimed by this study, the difference in $\Sigma^2(r)$ behavior arises due to the difference in large $r$ ($\Lambda \gg r \gg \sqrt{\Lambda}$), behavior of $R_2(r)$ in the two cases.

As our technique is equally applicable to both these systems, multifractality should exist in either both or none of them. Note that the multifractal nature of an ensemble is so far believed to be indicated by its $\Sigma^2$ behavior. But the latter is not yet clearly understood for the RP model (see [39,40]) and therefore the question of multifractality is still not fully settled. Also note that the earlier results for both models are obtained by the SUSY technique using the saddle-point approximation at various stages which may also be misleading. It is also possible that (i) the seeming multifractality of ensemble (44) is just the erroneous conclusion of too many approximations in the SUSY technique, or (ii) $\Sigma^2(r) \approx \chi r$ is not always a correct indicator of multifractality and therefore its absence in the RP model.

We believe that the $\Sigma^2(r)$ behavior is a bigger suspect than the approximations used in other techniques [33,41]. Our belief has its roots in the direct applicability of our technique to the Anderson model too. Here also the ensemble for $H$ is located between Poisson $\rightarrow$ GUE (for a time-reversal breaking disorder) with corresponding $R_2$ behavior given by Eq. (38). Thus for finite $\Lambda$ values corresponding to the critical region, the eigenvalue statistics is different from Poisson or GUE. But again for $\Sigma^2$ obtained by using Eq. (38), $\Sigma^2(r) \neq \chi r$ with $\chi < 1$ and therefore if it is indeed an indicator of multifractality of eigenfunctions, our technique would suggest its absence in the Anderson model. However, the existence of multifractality among the eigenfunction of the Anderson Hamiltonian is experimentally confirmed.

Our results indicate that multifractality either will be a common feature of all the Gaussian ensembles with finite $\Lambda$ values in the limit $N \rightarrow \infty$ or it does not exist in any of them. Thus the questions related to critical statistics, the correct criteria for multifractality, and its analysis by the SUSY technique require further probing.

## IV. OTHER CASES

### A. A perturbed Hamiltonian with *GG*-type perturbation

The intimate connection of the RM theory$\rightarrow$CM Hamiltonian continues also for system $H = H_0 + xV$ with a random perturbation $V$ drawn from a GGE [i.e., $\rho(V,y,b) = C \exp(-\Sigma_{s=1}^{2} \Sigma_{k \leqslant l} \alpha_{kl;s} (V_{kl;s} - b_{kl;s})^2$]. In this case, the eigenvalues evolve due to the changing strength of the perturbation too. To obtain the desired evolution equation, therefore, one needs to consider the sum

$$\frac{\partial P}{\partial x} + \sum_s \sum_{k \leqslant l} (\gamma - y_{kl;s}) \left[ 2 y_{kl;s} \frac{\partial P}{\partial y_{kl;s}} - b_{kl;s} \frac{\partial P}{\partial b_{kl;s}} \right],$$

which leads to the following equality:

$$\frac{\partial P}{\partial x} + \sum_s \sum_{k \leqslant l} (\gamma - y_{kl;s}) \left[ 2 y_{kl;s} \frac{\partial P}{\partial y_{kl;s}} - b_{kl;s} \frac{\partial P}{\partial b_{kl;s}} \right] = \sum_s \sum_{k \leqslant l} I_{kl;s} - \sum_s \sum_{k \leqslant l} y_{kl;s} I_{kl;s}, \tag{49}$$

where $I_{kl;s}$ is still given by the same form as Eq. (12) but with $H$ replaced by $V$. As the right-hand side of Eq. (49) is the same as that of Eq. (14), one again obtains the evolution equation (17) but now $Y$ is given by the condition that

$$\frac{\partial P}{\partial Y} = \frac{\partial P}{\partial x} + \sum_s \sum_{k \leqslant l} (\gamma - y_{kl;s}) \left[ 2 y_{kl;s} \frac{\partial P}{\partial y_{kl;s}} - b_{kl;s} \frac{\partial P}{\partial b_{kl;s}} \right].$$

Proceeding just as in Sec. I C, $Y$ can be shown to be given by the following relation:

$$Y = \frac{1}{2N^2 + 1} \left[ x + \sum_{k \leqslant l} \sum_{s=1}^{2} \left( \frac{1}{2} \ln \frac{y_{kl;s}}{|y_{kl;s} - \gamma|} - \gamma^{-1} \ln |b_{kl;s}| \right) \right] + C. \tag{50}$$

Again the steady state is achieved for $Y \rightarrow \infty$ which corresponds to $x \rightarrow \infty$ and $y_{kl;s} \rightarrow \gamma$; the steady-state solution for $P$ is given by $\Pi_{i<j} |\mu_i - \mu_j|^{\beta} e^{-(\gamma/2) \Sigma_k \mu_k^2}$. Here only $x \rightarrow \infty$ (with $\partial P / \partial x = 0$ and $H = xV$) no longer represents a steady state, as in the case when $V$ belongs to SGE, but it represents a transition state with $\partial P / \partial Y \neq 0$. Note from Eq. (50) that $Y \rightarrow \infty$ as $x \rightarrow \infty$, seemingly implying that the equilibrium is reached and therefore $H$ belongs to SGE. But, as obvious from $H = H_0 + xV$, in the limit $x \rightarrow \infty$, $H = xV$ and therefore $H$ must be a GG matrix. This contradiction is a result of the error made in not ensuring the mean spacing of $H$ the same as $H_0$ and $V$ [6]. Here, to ensure the latter, we need to use a modified Hamiltonian, given by $H = e^{-\tau/N} H_0 + \sqrt{(1 - e^{-2\tau/N})/N} V$ with $\tau = -N^{-1} \ln \cos(x/N)$ (the same as before in the large-$N$ limit). The effect of this modification on the $FP$ equation (17) is that now

$$\frac{\partial P}{\partial Y} = \frac{\partial P}{\partial \tau} + \frac{1}{N(1 - e^{-2\tau/N})} \sum_s \sum_{k \leqslant l} (\gamma - y_{kl;s}) \times \left[ 2 y_{kl;s} \frac{\partial P}{\partial y_{kl;s}} - \gamma^{-1} b_{kl;s} \frac{\partial P}{\partial b_{kl;s}} \right]$$

and the coefficient $\gamma$ of the drift term is now replaced by $N^{-1} \gamma$ [see Eq. (13) of [42]]. The $Y$ can now be obtained by the second method given in Sec. I C.

### B. Non-Gaussian ensembles

As mentioned before, the RM models of complex systems can, in general, be non-Gaussian, e.g., $\rho(H) = C \exp[-\Sigma_{k \leqslant l} f_{kl}(H_{kl})]$ with $f$ as an arbitrary function, and it is not an easy task to obtain the correlations in this case. However, this case can be analyzed by our method if $f$ is a well behaved function and can be expanded in a Taylor's series. To understand this, let us consider an ensemble of real symmetric matrices $H$ with a distribution of a more general nature, e.g., $f$ as a polynomial of $H$ with degree $2M$, $f_{kl}(x) = \Sigma_{r=1}^{M} \gamma_{kl}(r) x^{2r}$ with $C$ as the normalization constant and variances for the diagonal and off-diagonal matrix elements chosen to be arbitrary.

To obtain an evolution equation in this case, we now consider the sum

$$2\sum_{r=1}^{M} r\sum_{k\leqslant l}[\gamma-y_{kl}(1)]y_{kl}(r)\frac{\partial\tilde{P}}{\partial y_{kl}(r)}$$

[with $P=C\tilde{P}$ and $y_{kl}(r)=g_{kl}\gamma_{kl}(r)$] where the derivative of $\tilde{P}$ with respect to $\gamma_{kl}(1)$ can be shown to be the following (with $\rho=C\tilde{\rho}$):

$$\frac{\partial\tilde{P}}{\partial\gamma_{kl}(1)}=\frac{1}{2\gamma_{kl}(1)}\int\prod_{i=1}^{N}\delta(\mu_i-\lambda_i)H_{kl}\frac{\partial\tilde{\rho}}{\partial H_{kl}}dH - \sum_{r=2}^{M} r\frac{\gamma_{kl}(r)}{\gamma_{kl}(1)}\int\prod_{i=1}^{N}\delta(\mu_i-\lambda_i)\frac{\partial\tilde{\rho}}{\partial\gamma_{kl}(r)}dH. \tag{51}$$

Now as $\partial\tilde{\rho}/\partial\gamma_{kl}(r)=-H_{kl}^{2r}\rho$ and $\partial\tilde{\rho}/\partial H_{kl}=-2\Sigma_{r=1}^{M}r\gamma(r)H_{kl}^{2r-1}\rho$, the second integral in Eq. (51) being equal to $\partial\tilde{P}/\partial\gamma_{kl}(r)$, Eq. (11) can be rearranged to show that $2\Sigma_{r=1}^{M}r\gamma_{kl}(r)(\partial\tilde{P}/\partial\gamma_{kl}(r))=I_{kl}$ with $I_{kl}$ given by Eq. (12), [but without subscript ($s$) on quantities]. The required evolution equation in this case can be obtained from the following equality:

$$2\sum_{r=1}^{M}\sum_{k\leqslant l} r[\gamma-y_{kl}(1)]y_{kl}(r)\frac{\partial\tilde{P}}{\partial y_{kl}(r)} = \gamma\sum_{k\leqslant l}I_{kl}-\sum_{k\leqslant l}y_{kl}(1)I_{kl} \tag{52}$$

where, again, $\Sigma_{k\leqslant l}I_{kl}=\Sigma_n(\partial/\partial\mu_n)(\mu_n\tilde{P})$ and

$$\sum_{k\leqslant l}y_{kl}(1)I_{kl}=-\sum_{n}\frac{\partial}{\partial\mu_n}\left[\frac{\partial}{\partial\mu_n}+\sum_{m\neq n}\frac{\beta}{\mu_m-\mu_n}\right]\tilde{P}+\sum_{k\leqslant l}J_{kl} \tag{53}$$

with $J_{kl}$ now given by the following relation:

$$J_{kl}=-\sum_{n}\frac{\partial}{\partial\mu_n}\sum_{k\leqslant l}\int\prod_{i=1}^{N}\delta(\mu_i-\lambda_i)\frac{\partial\lambda_n}{\partial H_{kl}} \times\left[\sum_{r=2}^{M}ry_{kl}(r)H_{kl}^{2r-1}\right]\rho dH \tag{54}$$

$$=g_{kl}\sum_{r=1}^{M}(r+1)y_{kl}(r+1)\frac{\partial\tilde{P}}{\partial y_{kl}(r)}\left[(2r+1) +2\sum_{s=1}^{M}sy_{kl}(s)\frac{\partial\tilde{P}}{\partial y_{kl}(s)}\right]. \tag{55}$$

Using these relations as before, one again obtains the FP equation for $\tilde{P}$ similar to Eq. (17) with $\beta=1$ and $\partial\tilde{P}/\partial Y=2\Sigma_{k\leqslant l}\Sigma_{r=1}^{M}h_{kl}(r)[\partial\tilde{P}/\partial y_{kl}(r)]$ where

$$h_{kl}(r)=2ry_{kl}(r)[\gamma-y_{kl}(1)]+(r+1)(2r+1) \times y_{kl}(r+1)g_{kl}+2(r+1)y_{kl}(r+1)g_{kl} \times\Sigma_{s=1}^{M}sy_{kl}(s)[\partial\tilde{P}/\partial y_{kl}(r)].$$

Note that the condition for $Y$ here includes terms of type $[\partial\tilde{P}/\partial y_{kl}(r)][\partial\tilde{P}/\partial y_{kl}(s)]$ and $Y$ can no longer be obtained by methods given in Sec. I C.

### C. Block-diagonal ensembles

Equation (7) and, therefore, evolution equation (17) of $P(\mu,Y)$ is no longer valid if the matrix $H$ is in a block-diagonal form. This is because the eigenvalues belonging to different blocks do not repel each other, are not correlated, and undergo an evolution independent of the other block. For this case, the evolution of eigenvalues in each block can be considered separately, leading to one FP equation similar to Eq. (17) for each block. A detailed discussion of this case is given in [14].

## V. AN ALTERNATIVE EVOLUTION EQUATION FOR THE EIGENVALUES

In Sec. I B, the equation (17) governing the evolution of the eigenvalues was obtained by using the relation (14). However, as obvious from Eq. (13), $P$ also satisfies the relation

$$\sum_{k\leqslant l}\left[2y_{kl;s}\frac{\partial P}{\partial y_{kl;s}}-b_{kl;s}\frac{\partial P}{\partial b_{kl;s}}\right]=\sum_{k\leqslant l}I_{kl;s} \tag{56}$$

and, therefore, one can define a function $Z(y_{kl;s},b_{kl;s})$ such that

$$\frac{\partial P}{\partial Z}=\sum_{n}\frac{\partial}{\partial\mu_n}(\mu_nP). \tag{57}$$

Here $Z$ is given by the condition

$$\frac{\partial P}{\partial Z}=\sum_{k\leqslant l}\left[2y_{kl;s}\frac{\partial P}{\partial y_{kl;s}}-b_{kl;s}\frac{\partial P}{\partial b_{kl;s}}\right]$$

which can be solved (as in Sec. II) to show that

$$Z=\frac{1}{4N^2}\ln\left[\prod_{k\leqslant l}\prod_{s=1}^{2}|y_{kl;s}|b_{kl;s}^{-2}\right]+C.$$

Equation (57) also describes the evolution of eigenvalues for the same ensemble (3). But now the ''time'' scale is such that the eigenvalues seem to be drifting only, hiding the repulsion between them. Again the steady state of Eq. (57) is given by $|Z-Z_0|\to\infty$ and the final ensemble as Poisson (with finite, nonzero variances for diagonal matrix elements and zero variances for the off-diagonal ones). The ensemble $G$ will now appear as an intermediate point in a transition from some initial ensemble to the Poisson ensemble and, in principle, the transition can be used for the analysis of $G$. For example, the critical parameter for the Anderson transition (the same model as used in Sec. III) can be obtained by

taking the initial state ''$O$'' as metal with the energy-level distribution described by a GUE ($\langle O_{ii}^2\rangle=\alpha_M^{-1}$, $\langle O_{ij}^2\rangle=0$) and all $\langle O_{ij;s}\rangle=t_M$ which gives $Z_0=(1/2N^2)[N\ln\alpha_M-2K\ln|t_M|]+C$. The critical region will therefore occur as an intermediate point in the GUE $\rightarrow$ Poisson transition with transition parameter

$$\Lambda=D^{-2}(Z-Z_0)=\frac{1}{2N}\left[N\ln\frac{\alpha}{\alpha_M}-2K\ln\frac{|t|}{|t_M|}\right].$$

As is obvious, the increase of the diagonal disorder ($\alpha/\alpha_M<1$) for a fixed hopping rate ($t=t_M$) will ultimately lead to Poisson statistics, implying localization of states; note here the transition occurs backwards in ''time'' $\Lambda$. However, the results for correlations associated with the SGE $\rightarrow$ Poisson transition are inconclusive about which leaves Eq. (17) as a better tool to analyze the properties of GGEs. Equation (17) has one more advantage over Eq. (58): the reduction of the former to the CM Hamiltonian reveals the underlying universality of statistical formulation among various complex systems.

## VI. CONCLUSION

In this paper we have described a method to analyze the statistical properties of the RM model of complex systems. Our technique is based on the exact reduction of spectral analysis in the general case to the one in SGE. This greatly reduces the degree of difficulty of the original problem as many of the properties of SGE are already known. This also indicates that a thorough knowledge of the properties of SGE or CSM will be highly advantageous even for systems with interactions too intricate to be modeled by SGE. So far, the probing of GGE is carried out only by the SUSY technique which requires a saddle-point approximation at various steps and is not easily applicable, even approximately, to cases where our technique can be used for exact probing. Note that the main term in GGEs responsible for the correspondence with the CSM Hamiltonian is due to the repulsion between eigenvalues. As the mathematical origin of this term lies in the transformation from matrix space to eigenvalue space, which is the same for all the Hermitian ensembles (belonging to the same symmetry class), the correspondence with the CSM Hamiltonian should exist for almost all of them, irrespective of the distribution of their matrix elements. As discussed in Sec. III, our study also confirms the conjecture regarding the one parameter scaling of localization and provides the formula for a relevant parameter.

The reduction technique presented here raises some basic questions. Why does the reparametrization of the spectral properties of different RM ensembles result in a similar mathematical formulation for them? In other words, why do the eigenvalues of quantum operators associated with complex systems evolve in a similar ordered way (like equations of motion for Calogero particles) notwithstanding the varied nature of their complexity? The reason may lie in the following. The eigenvalues and eigenfunctions of a Hamiltonian evolve due to a change in either the degree or nature of its complexity. The evolution of the eigenfunction is chaotic in the sense that the overlapping between the eigenfunctions, associated with two Hamiltonians even with slightly different complexity, decreases rapidly in time (page 2 of [6]). However, an eigenvalue of an operator is its average value in the state described by the associated eigenfunction and an ordered evolution of the former will, in general, imply an ordered change in the average behavior of the latter. Thus it seems that the eigenvalues and eigenfunctions, on average, are not able to view the fine subtleties of the varied nature of complexity and therefore are not affected too drastically to loose correlations even when the nature of the complexity changes. Note, for a small change in the interactions, this result is not surprising and is used as the base for the perturbation theory. But the results in this paper imply that the eigenvalues (and the physics based on them) even after a violent change in the interactions remain correlated in the parametric space. Thus it seems that certain physical properties, based on the average behavior of eigenvalues and eigenfunctions, of one complex system are related to the physics of another, very different in nature of the interactions.

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## APPENDIX A: PROOF OF EQS. (3)–(5)

The use of the eigenvalue equation $HU=U\Lambda$, with $U$ as a unitary matrix and $\Lambda$ the eigenvalue matrix, leads to the following:

$$\sum_j H_{ij}U_{jn}=\lambda_n U_{in} \quad \text{and} \quad \sum_i H_{ij}U_{in}^*=\lambda_n U_{jn}^*, \tag{A1}$$

where $H_{ij}=H_{ij;1}+iH_{ij;2}$. Differentiating both sides of above equation with respect to $H_{kl;s}$ (with $s=1$ or 2), we get

$$\sum_j \frac{\partial U_{jn}}{\partial H_{kl;s}}H_{ij}+\sum_j U_{jn}\frac{\partial H_{ij}}{\partial H_{kl;s}}=\lambda_n\frac{\partial U_{in}}{\partial H_{kl;s}}+\frac{\partial\lambda_n}{\partial H_{kl;s}}U_{in}. \tag{A2}$$

Now as $\Sigma_i U_{in}^*U_{im}=\delta_{nm}$, multiplying both the sides by $U_{in}^*$ followed by a summation over all $i$'s, we get the following:

$$\frac{\partial\lambda_n}{\partial H_{kl;s}}=\sum_{i,j} U_{in}^*\frac{\partial H_{ij}}{\partial H_{kl;s}}U_{jn}, \tag{A3}$$

which further gives

$$\frac{\partial\lambda_n}{\partial H_{kl;s}}=i^{s-1}\frac{1}{g_{kl}}[U_{ln}U_{kn}^*-(-1)^s U_{ln}^*U_{kn}]. \tag{A4}$$

This can further be used to show that

$$\sum_{k\leqslant l}\sum_{s=1}^{2}\frac{\partial\lambda_n}{\partial H_{kl;s}}H_{kl;s}=\sum_{k\leqslant l}\frac{1}{g_{kl}}\Big[U_{ln}U_{kn}^{*}\sum_{s}\ i^{s-1}H_{kl;s}+U_{ln}^{*}U_{kn}\sum_{s}\ i^{s-1}(-1)^{s+1}H_{kl;s}\Big] \tag{A5}$$

$$=\sum_{k\leqslant l}\frac{1}{g_{kl}}[H_{kl}U_{ln}U_{kn}^{*}+H_{kl}^{*}U_{ln}^{*}U_{kn}] \tag{A6}$$

$$=\sum_{k\leqslant l}\frac{1}{g_{kl}}H_{kl}U_{ln}U_{kn}^{*}+\sum_{k\geqslant l}\frac{1}{g_{kl}}H_{lk}^{*}U_{kn}^{*}U_{ln} \tag{A7}$$

$$=\sum_{k,l}\ H_{kl}U_{ln}U_{kn}^{*}=\lambda_n, \tag{A8}$$

where Eq. (A8) is obtained from Eq. (A7) by using Hermitian properties of $H$ ($H_{lk}^{*}=H_{kl}$). By using Eq. (A4), one can also show that

$$\sum_{k\leqslant l}g_{kl}\sum_{s=1}^{2}\frac{\partial\lambda_n}{\partial H_{kl;s}}\frac{\partial\lambda_m}{\partial H_{kl;s}}$$

$$=\sum_{k\leqslant l}\sum_{s=1}^{2}\ i^{2(s-1)}\frac{1}{g_{kl}}[U_{ln}U_{kn}^{*}-(-1)^{s}U_{ln}^{*}U_{kn}]\times[U_{lm}U_{km}^{*}-(-1)^{s}U_{lm}^{*}U_{km}] \tag{A9}$$

$$=\sum_{k\leqslant l}\frac{2}{g_{kl}}[U_{ln}U_{kn}^{*}U_{km}U_{lm}^{*}+U_{ln}^{*}U_{kn}U_{lm}U_{km}^{*}] \tag{A10}$$

$$=2\sum_{k,l}\ U_{ln}U_{kn}^{*}U_{km}U_{lm}^{*}$$

$$=\sum_{k}\ U_{kn}U_{km}^{*}\sum_{l}\ U_{lm}U_{ln}^{*}$$

$$=2\,\delta_{mn}, \tag{A11}$$

where Eq. (A11) follows from Eq. (A10) by writing $\Sigma_{k\leqslant l}U_{ln}^{*}U_{kn}U_{km}^{*}U_{lm}=\Sigma_{k\geqslant l}U_{ln}U_{kn}^{*}U_{km}U_{lm}^{*}$ and the last equality in Eq. (A11) is due to unitary nature of $U$.

## APPENDIX B: PROOF OF EQ. (6)

Multiplying both the sides of Eq. (A2) by $U_{im}^{*}$ ($m\neq n$) followed by a summation over all $i$'s, we get the following:

$$\sum_{j}\ U_{jm}^{*}\frac{\partial U_{jn}}{\partial H_{kl;s}}=\frac{1}{\lambda_n-\lambda_m}\sum_{i,j}\ U_{im}^{*}\frac{\partial H_{ij}}{\partial H_{kl;s}}U_{jn}, \tag{B1}$$

a multiplication of both the sides by $U_{rm}$ followed by a summation over all $m$'s then gives

$$\frac{\partial U_{rn}}{\partial H_{kl;s}}=i^{s-1}\frac{1}{g_{kl}}\sum_{m\neq n}\frac{U_{rm}}{\lambda_n-\lambda_m}[U_{km}^{*}U_{ln}-(-1)^{s}U_{lm}^{*}U_{kn}]. \tag{B2}$$

## APPENDIX C: PROOF OF EQ. (7)

$$\sum_{k\leqslant l}g_{kl}\sum_{s=1}^{2}\frac{\partial^2\lambda_n}{\partial H_{kl;s}^{2}}=\sum_{k\leqslant l}\sum_{s=1}^{2}\ i^{s-1}\frac{1}{g_{kl}}\frac{\partial}{\partial H_{kl;s}}\times[U_{ln}U_{kn}^{*}-(-1)^{s}U_{ln}^{*}U_{kn}] \tag{C1}$$

$$=\sum_{k\leqslant l}\sum_{s=1}^{2}\ i^{s-1}\Big[\frac{\partial U_{kn}^{*}}{\partial H_{kl;s}}U_{ln}+\frac{\partial U_{ln}}{\partial H_{kl;s}}U_{kn}^{*}+(-1)^{s+1}\frac{\partial U_{ln}^{*}}{\partial H_{kl;s}}U_{kn}+(-1)^{s+1}\frac{\partial U_{kn}}{\partial H_{kl;s}}U_{ln}^{*}\Big]. \tag{C2}$$

Now by using Eq. (B2) and its complex conjugate in Eq. (C2) and by summing over $s$, we get

$$\sum_{k\leqslant l}g_{kl}\sum_{s=1}^{2}\frac{\partial^2\lambda_n}{\partial H_{kl;s}^{2}}=4\sum_{k\leqslant l}\frac{1}{g_{kl}}\sum_{m}\frac{1}{\lambda_n-\lambda_m}[U_{km}U_{km}^{*}U_{ln}U_{ln}^{*}+U_{kn}U_{kn}^{*}U_{lm}U_{lm}^{*}] \tag{C3}$$

$$=4\sum_{k,l}\ \sum_{m}\frac{1}{\lambda_n-\lambda_m}[U_{km}U_{km}^{*}U_{ln}U_{ln}^{*}] \tag{C4}$$

$$=4\sum_{m}\frac{1}{\lambda_n-\lambda_m}\Big[\sum_{k}\ U_{km}U_{km}^{*}\Big]\times\Big[\sum_{l}\ U_{ln}U_{ln}^{*}\Big]. \tag{C5}$$

Now by using the unitary relation $\Sigma_j U_{jm}^{*}U_{jm}=1$, one obtains the desired relation (7).

## APPENDIX D: PROOF OF EQ. (15)

Equation (12) gives us the following:

$$\sum_{k\leqslant l}\sum_{s=1}^{2}\ I_{kl;s}=\sum_{n}\frac{\partial}{\partial\mu_n}\int\prod_{i}\ \delta(\mu_i-\lambda_i)\times\Big[\sum_{k\leqslant l}\sum_{s=1}^{2}\frac{\partial\lambda_n}{\partial H_{kl;s}}H_{kl;s}\Big]\rho dH. \tag{D1}$$

The use of Eq. (A8) will further simplify it in following form:

$$\sum_{k\leqslant l}\sum_{s=1}^{2}\ I_{kl;s}=\sum_{n}\frac{\partial}{\partial\mu_n}\int\prod_{i}\ \delta(\mu_i-\lambda_i)\lambda_n\rho dH \tag{D2}$$

$$=\sum_{n}\frac{\partial}{\partial\mu_n}(\mu_n P). \tag{D3}$$

## APPENDIX E: PROOF OF EQ. (16)

For each $s$ value, we have the following relation:

$$\sum_{k\leqslant l} y_{kl;s} I_{kl;s} = \sum_{n=1}^{N} \frac{\partial}{\partial \mu_n} \sum_{k\leqslant l} g_{kl}\alpha_{kl;s} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \lambda_n}{\partial H_{kl;s}} H_{kl;s} \rho dH \tag{E1}$$

$$= -\sum_{n=1}^{N} \frac{\partial}{\partial \mu_n} \sum_{k\leqslant l} \frac{g_{kl}}{2} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \lambda_n}{\partial H_{kl;s}} \left[ \frac{\partial}{\partial H_{kl;s}} - 2\alpha_{kl;s} b_{kl;s} \right] \rho dH \tag{E2}$$

$$= -\sum_{n=1}^{N} \frac{\partial}{\partial \mu_n} \sum_{k\leqslant l} \frac{g_{kl}}{2} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \lambda_n}{\partial H_{kl;s}} \frac{\partial \rho}{\partial H_{kl;s}} dH + \sum_{k\leqslant l} J_{kl;s}, \tag{E3}$$

where Eq. (E3) is obtained by using the equality $\partial\rho/\partial H_{kl;s} = -2\alpha_{kl;s}(H_{kl;s} - b_{kl;s})\rho$ and $J_{kl;s}$ is given by Eq. (E9).

By integrating Eq. (E3) further by parts, one obtains

$$\sum_{k\leqslant l} \sum_{s=1}^{2} y_{kl;s} I_{kl;s} = \sum_{s} \sum_{n} \frac{\partial}{\partial \mu_n} \sum_{k\leqslant l} \frac{g_{kl}}{2} \int \left( \frac{\partial}{\partial H_{kl;s}} \prod_{i} \delta(\mu_i - \lambda_i) \right) \frac{\partial \lambda_n}{\partial H_{kl;s}} \rho dH \tag{E4}$$

$$+ \sum_{s} \sum_{n} \frac{\partial}{\partial \mu_n} \sum_{k\leqslant l} \frac{g_{kl}}{2} \int \prod_{i} \delta(\mu_i - \lambda_i) \frac{\partial^2 \lambda_n}{\partial H_{kl;s}^2} \rho dH + \sum_{k\leqslant l} \sum_{s} J_{kl;s} \tag{E5}$$

$$= -\sum_{n} \frac{\partial}{\partial \mu_n} \sum_{m} \frac{\partial}{\partial \mu_m} \int \prod_{i} \delta(\mu_i - \lambda_i) \left[ \sum_{s} \sum_{k\leqslant l} \frac{g_{kl}}{2} \frac{\partial \lambda_n}{\partial H_{kl;s}} \frac{\partial \lambda_m}{\partial H_{kl;s}} \right] \rho dH \tag{E6}$$

$$- \sum_{n} \frac{\partial}{\partial \mu_n} \int \prod_{i} \delta(\mu_i - \lambda_i) \left[ \sum_{m\neq n} \frac{2}{\lambda_m - \lambda_n} \right] \rho(H) dH + \sum_{k\leqslant l} \sum_{s} J_{kl;s} \tag{E7}$$

$$= -\sum_{n} \frac{\partial^2 P}{\partial \mu_n^2} - \sum_{n} \frac{\partial}{\partial \mu_n} \left[ 2 \sum_{m\neq n} \frac{P}{\mu_m - \mu_n} \right] + \sum_{k\leqslant l} \sum_{s} J_{kl;s}, \tag{E8}$$

where $J_{kl;s}$ can be obtained as follows:

$$J_{kl;s} = y_{kl;s} b_{kl;s} \sum_{n=1}^{N} \frac{\partial}{\partial \mu_n} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \lambda_n}{\partial H_{kl;s}} \rho dH \tag{E9}$$

$$= -y_{kl;s} b_{kl;s} \int \partial \prod_{i=1}^{N} \frac{\delta(\mu_i - \lambda_i)}{\partial H_{kl;s}} \rho dH \tag{E10}$$

$$= y_{kl;s} b_{kl;s} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \rho}{\partial H_{kl;s}} dH \tag{E11}$$

$$= -y_{kl;s} b_{kl;s} \int \prod_{i=1}^{N} \delta(\mu_i - \lambda_i) \frac{\partial \rho}{\partial b_{kl;s}} dH = -y_{kl;s} b_{kl;s} \frac{\partial P}{\partial b_{kl;s}}, \tag{E12}$$

where in Eq. (A32), the equality $\partial\rho/\partial b_{kl;s} = 2\alpha_{kl;s}(H_{kl;s} - b_{kl;s})\rho = -\partial\rho/\partial H_{kl;s}$ is used. A substitution of Eq. (E12) in Eq. (E8) now leads to the Eq. (16).

## APPENDIX F: A GENERAL METHOD OF OBTAINING $Y$

Let us consider a transformation of $M = 2N^2$ coordinates $\{r_j\}$ to another set of $M$ coordinates $\{Y_i\}$, where $r_j$'s are various coefficients $y_{kl;s}$ (total $N^2$) and $b_{kl;s}$ (total $N^2$). The $Y_i$'s should be chosen such that the right-hand side of Eq. (18), summing over all $y_{kl;s}$,'s and $b_{kl;s}$'s, can be rewritten as

$$\sum_{i}^{M} \frac{\partial P}{\partial Y_i} = \sum_{k\leqslant l} 2(\gamma - y_{kl;s}) y_{kl;s} \frac{\partial P}{\partial y_{kl;s}} - \gamma \sum_{k\leqslant l} b_{kl;s} \frac{\partial P}{\partial b_{kl;s}}$$

$$\equiv \sum_{j=1}^{M} g_j(r_1, r_2, \ldots, r_M) \frac{\partial P}{\partial r_j}, \tag{F1}$$

where, for our case, $g_i(r_1, \ldots, r_M) = 2(\gamma - r_i) r_i$ if $r_i$ is one

of the $y_{kl;s}$ and $g_i(r_1,\ldots,r_M)=-\gamma r_i$ if $r_i$ is one of the $b_{kl;s}$.

Now, as we want $\Sigma_i^M(\partial/\partial Y_i)=(\partial/\partial Y_1)$, with $Y_1\equiv Y$, this imposes the following conditions on the functions $Y_i$'s (as can be shown by using the theory of partial differentiation):

$$\frac{\partial P}{\partial Y_1}=\sum_{i=1}^{M}\sum_{j}^{M} g_j(r_1,r_2,\ldots,r_M)\frac{\partial P}{\partial Y_i}\frac{\partial Y_i}{\partial r_j}, \tag{F2}$$

and therefore

$$\sum_{j=1}^{M} g_j(r_1,r_2,\ldots,r_M)\frac{\partial Y_i}{\partial r_j}=\delta_{1i}. \tag{F3}$$

According to the theory of partial differential equations [15], the general solution of linear PDE $\Sigma_i^M P_i(x_1,x_2,\ldots,x_M)(\partial Z/\partial x_i)=R$ is $F(u_1,u_2,\ldots,u_n)=0$, where $F$ is an arbitrary function and $u_i(x_1,x_2,\ldots,x_n,Z)=c_i$ (a constant), $i=1,2,\ldots,n$ are independent solutions of the following equation:

$$\frac{dx_1}{P_1}=\frac{dx_2}{P_2}=\cdots\frac{dx_k}{P_k}=\cdots\frac{dx_M}{P_M}=\frac{dZ}{R}. \tag{F4}$$

Thus the general solution of Eq. (F3) for each $Y_j$ is given by a relation $F_j(u_{1j},u_{2j},\ldots,u_{Mj})=0$ where function $F_j$ is arbitrary and $u_{ij}(r_1,r_2,\ldots,r_M,Y_j)=c_{ij}$, $(i=1,2,\ldots,M)$ (with $c_{ij}$'s as constants) are independent solutions of the equation

$$\frac{dr_1}{g_1}=\frac{dr_2}{g_2}=\cdots\frac{dr_k}{g_k}=\cdots\frac{dr_M}{g_M}=\frac{dY_j}{\delta_{1j}}. \tag{F5}$$

The above set of equations can be solved for various $Y_j$ to obtain $F_j$. For $Y_1$, we get the relations $Y_1-\frac{1}{2}\log(r_i/|r_i-\gamma|)=c_{i1}$ $(i=1,\ldots,M/2)$, $Y_1+(1/\gamma)\log|r_i|=c_{i1}$ $(i=1+M/2,\ldots,M)$, and therefore $F_1$ satisfies the relation $F_1[Y_1-\frac{1}{2}\log(r_1/|r_1-\gamma|),\ldots,Y_1-\frac{1}{2}\log(r_{M/2}/|r_{M/2}-\gamma|),Y_1+\gamma^{-1}\log|r_{M/2}|,\ldots,Y_1+\gamma^{-1}\log|r_M|]=0$. The function $F_1$ being arbitrary here, this relation can also be expressed in the following form:

$$Y_1=\frac{1}{M}\left[\frac{1}{2}\sum_{i=1}^{M/2}\log\frac{r_i}{|r_i-\gamma|}-\frac{1}{\gamma}\sum_{i=M/2+1}^{M}\log|r_i|\right]+C, \tag{F6}$$

where $C$ is another arbitrary function of constants: for example,

$$C\equiv C[\tfrac{1}{2}\log(r_1/|r_1-\gamma|)+\gamma^{-1}\log|r_M|,$$

$$\tfrac{1}{2}\log(r_2/|r_2-\gamma|)+\gamma^{-1}\log|r_M|,\ldots,$$

$$\tfrac{1}{2}\log(r_{M-1}/|r_{M-1}-\gamma|)+\gamma^{-1}\log|r_M|].$$

Similarly the variables $Y_i$, $i>1$ can be obtained; however, their knowledge is not required for our analysis.

## APPENDIX G

The choice of $\gamma$ is based only on the requirement that $y_{kl}(O)>y_{kl}(G)>\gamma$ for all $k,l$. Thus $\gamma$ can take any value such that $\gamma\leqslant\min y_{kl}(G)$. Let us consider two such possibilities for $\gamma$, $\gamma=\gamma_1$ and $\gamma=\gamma_2$ and try to evaluate properties of $G$ on these curves referred to as $T1$ and $T2$, respectively (for some initial ensemble). Let the value of $Y$ for $G$ on these curves be $Y_1$ and $Y_2$ where

$$Y_1=\frac{1}{2N^2}\sum_{k\leqslant l}\sum_{s=1}^{2}\left[\frac{1}{2}\ln\frac{y_{kl;s}}{|y_{kl;s}-\gamma_1|}-\frac{1}{\gamma_1}\ln b_{kl;s}\right]+C, \tag{G1}$$

$$Y_2=\frac{1}{2N^2}\sum_{k\leqslant l}\sum_{s=1}^{2}\left[\frac{1}{2}\ln\frac{y_{kl;s}}{|y_{kl;s}-\gamma_2|}-\frac{1}{\gamma_2}\ln b_{kl;s}\right]+C. \tag{G2}$$

However, $Y_1$ can also be written as follows:

$$Y_1=\frac{1}{2N^2}\sum_{k\leqslant l}\sum_{s=1}^{2}\left[\frac{1}{2}\ln\frac{y'_{kl;s}}{|y'_{kl;s}-\gamma_2|}-\frac{1}{\gamma_2}\ln b'_{kl;s}\right]+C. \tag{G3}$$

Now as $y'_{kl;s}=y_{kl;s}(\gamma_2/\gamma_1)\neq y_{kl;s}$ and $\gamma_2/\gamma_1 b'_{kl;s}=b$ this implies that $Y_1$ would correspond to a point, different from $Y_2$, on the transition curve $T_2$ and therefore would give properties for the ensemble $G$ different from those given by $Y_2$. This conclusion is, however, erroneous and is a result of the rescaling applied only to one point $Y_1$ on the transition curve $T_1$. To get the right answer, the entire curve $T_1$ should be rescaled which would also require a rescaling of the endpoints and therefore changed distances on the rescaled curve (call it $T'_1$). Thus the point $Y_1$ will appear at the same location on the $T'_1$ curve, relative to endpoints, where $Y_2$ appears on the $T_2$ curve and therefore both will imply the same properties for the ensemble $G$.


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